III. Solution by the PROPOSER.

Let $ABCD$ be the given square whose side$=a$, and MN one of the straight lines within the square. Draw AE equal and parallel to MN, EF parallel to AB cutting BC in F, and EG parallel to AD cutting CD in G. Put $MN=AE=x$ and $\angle EAD=\theta$. If x and θ were fixed the number of lines of the length a would equal the number of points in the rectangle $EFCG$ whose area is $a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta$. Then if all lines are supposed to be equally distributed about the point M the required average is

$$A_1=\frac{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]xdx}{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]dx}$$

$$=\frac{\frac{a^4}{12}\int_0^{\frac{1}{4}\pi}(2-\tan\theta)\sec^2\theta\, d\theta}{\frac{a^3}{6}\int_0^{\frac{1}{4}\pi}(3-\tan\theta)\sec\theta\, d\theta}=\frac{3a}{4[\log(1+\sqrt{2})^3+1-\sqrt{2})}.$$

If the lines are supposed to be so distributed as to join every possible pair of points in the square, the required average is

$$A_2=\frac{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]x^2dx}{\int_0^{\frac{1}{4}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]xdx}$$

$$=\frac{\frac{a^5}{60}\int_0^{\frac{1}{4}\pi}(5-3\tan\theta)\sec^3\theta d\theta}{\frac{a^4}{12}\int_0^{\frac{1}{4}\pi}(2-\tan\theta)\sec^2\theta d\theta}=\frac{a}{15}[2+\sqrt{2}+5\log(1+\sqrt{2})].$$

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

*Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let a, b, c be the sides; A, B, C the angles of the triangle; u, v variable lengths from A on b, c; w, x variable lengths from B on a, c; y, z variable lengths from C on a, b. Also let $l_1=\sqrt{(u^2+v^2-2uv\cos A)}$, $l_2=\sqrt{(w^2+x^2-2wx\cos B)}$, $l_3=\sqrt{(y^2+z^2-2yz\cos C)}$, M=average length. Then

*Another solution of this problem will be published next month.

$$M=\frac{\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} l\,dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} l\,dy\,dz\,dl}{\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} dy\,dz\,dl}$$

$$=N/D.$$

$$\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl=\tfrac{1}{2}\int_0^b\int_0^c(u^2+v^2-2uv\cos A)du\,dv$$

$$=\tfrac{1}{6}\int_0^b(3u^2c+c^3-3uc^2\cos A)du=\frac{bc}{12}(2b^2+2c^2-3bc\cos A).$$

$$\therefore N=\frac{2abc(a+b+c)+a^2b^2\cos C+a^2c^2\cos B+b^2c^2\cos A}{12}.$$

This follows, because $\frac{bc}{12}(2b^2+2c^2-3bc\cos A)=\frac{bc}{12}(2a^2+bc\cos A)$.

$$\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl=\int_0^k\int_0^c\sqrt{(u^2+v^2-2uv\cos A)}du\,dv$$

$$=\tfrac{1}{2}\int_0^b\Big[u^2\cos A+(c-u\cos A)\sqrt{(u^2+c^2-2uc\cos A)}$$

$$+u^2\sin^2 A\log\left(\frac{c-u\cos A+\sqrt{(u^2+c^2-2uc\cos A)}}{u(1-\cos A)}\right)\Big]du$$

$$=\tfrac{1}{6}(b^3+c^3-a^3)\cos A+\tfrac{1}{3}abc\sin^2 A+\tfrac{1}{6}b^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)$$

$$+\tfrac{1}{6}c^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}C).$$

$$\therefore D=\tfrac{1}{6}[(b^3+c^3-a^3)\cos A+(a^3+c^3-b^3)\cos B+(a^3+b^3-c^3)\cos C+2abc(\sin^2 A$$

$$+\sin^2 B+\sin^2 C)+b^3\sin^2 A\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)+a^3\sin^2 B\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B)$$

$$+(c^3\sin^2 A+a^3\sin^2 C)\log(\cot\tfrac{1}{2}A\cot\tfrac{1}{2}C)$$

$$+(b^3\sin^2 C+c^3\sin^2 B)\log(\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C)].$$

If the line is terminated by two of the sides

$$M_1=\frac{\int_0^b\int_0^c l_1\,du\,dv+\int_0^a\int_0^c l_2\,dw\,dx+\int_0^a\int_0^b l_3\,dy\,dz}{\int_0^b\int_0^c du\,dv+\int_0^a\int_0^c dw\,dx+\int_0^a\int_0^b dy\,dz}=\frac{D}{bc+ac+ab}.$$

Corollary. If $a=b=c$, $N=\frac{5}{8}a^4$, $D=a^3(1+\log 3)$.

$$\therefore M=\frac{5a}{8(1+\log 3)}. \quad M_1=\tfrac{1}{3}a(1+\log 3).$$

CALCULUS.

207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If K represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{K d\kappa}{1+\kappa}=\tfrac{1}{4}\pi^2.$$

Solution by S. A. COREY, Hiteman, Iowa.

As $K=\int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}$, the definite integral may be written

$$\int_0^1\int_0^1 \frac{dx\, d\kappa}{(1+\kappa)\sqrt{[(1-x^2)(1-\kappa^2x^2)]}},$$

$$\text{or} \int_0^1 \frac{1}{(1-x^2)^{\frac{1}{2}}}\left[\int_0^1 \frac{d\kappa}{(1+\kappa)(1-\kappa^2x^2)^{\frac{1}{2}}}\right]dx \ldots\ldots\ldots (1).$$

If $r=\kappa+1$, $\int_0^1 \frac{d\kappa}{(1+\kappa)\sqrt{(1-\kappa^2x^2)}}=\int_1^2 \frac{dr}{r\sqrt{X}}=$

$$-\left[\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x}+\frac{b}{2\sqrt{a}}\right)\right]_1^2=-\frac{1}{\sqrt{a}}\log\left(\frac{1}{1+\sqrt{a}}\right)=\frac{1}{\sqrt{a}}\log(1+\sqrt{a}),$$

in which $a=(1+x^2)$, $b=2x^2$, and $X=(1-x^2)+2x^2r-x^2r^2$.

The definite integral (1), after substituting, becomes,

$$\int_0^1 \frac{\log[1+\sqrt{(1-x^2)}]}{(1-x^2)}dx, \text{ or, if } x=\frac{1-z^2}{1+z^2}, \quad \int_0^1 \frac{1}{z}.\log\left(\frac{(1+z)^2}{1+z^2}\right)dz$$

$$=2\int_0^1 \frac{\log(1+z)}{z}dz-\int_0^1 \frac{\log(1+z^2)}{z}dz=2.\frac{\pi^2}{6}-\frac{\pi^2}{2.6}=\frac{\pi^4}{4}.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0=\frac{d}{dx}\left[(a^2+x^2)\frac{dy}{dx}\right].$$